I. Simple linear oscillators

A. Definitions and Background

Oscillators are a basic element in any measurement system and, in particular, are the core element in any clocks. By definition, an *oscillator* is a device that produces a signal that varies sinusoidally as a function of time, so that if s is the signal and t is time, we have $s(t) = s_0 \cos(\omega t + \phi_0)$, where s_0 and ϕ_0 are constants, and $\omega = f/2\pi$ is the angular frequency and f is the usual frequency, measured in Hz or s^{-1}. The signal can be a mechanical signal. Examples are the oscillations of a pendulum or a tuning fork. Typical frequencies for a pendulum clock are on the order of 1 Hz, while typical frequencies for a tuning fork are on the order of 500 Hz. The signal can be an electrical signal. Typical frequencies in this case range from a few kilohertz to hundreds of MHz. The signals can come from crystal vibrations. Quartz crystal oscillators, which are the workhorse of most timekeeping systems, oscillate at frequencies from a few kHz to tens of MHz. The best oscillators use atomic transitions. When an atom or molecule gives up energy, it emits a pure electromagnetic wave. The oscillations in this case are at much higher frequencies. For cesium atoms, which are the basis of modern-day atomic clocks, the frequency is close to 10×10^9 Hz or 10 GHz. Optical transitions occur at frequencies that are up to 100,000 times higher. Clocks are currently being developed at NIST that use a lattice of Yb atoms and that have an oscillation frequency of about 500×10^{12} Hz or 500 THz.

Operation at higher frequency is important because clocks are made by counting the number of zero crossings. A larger number of zero-crossings per second makes it possible to develop more accurate clocks.

Strictly speaking, an oscillator does not have to produce a sinusoidal signal. It is sufficient for the signal to repeat periodically, so that $s(t+T) = s(t)$, and in fact real oscillators do not produce strictly sinusoidal signals. Some *nonlinearity* is always present. However, it is a remarkable fact of nature that any oscillator, when it operates at a sufficiently low amplitude, will become sinusoidal. Moreover, any real oscillator will not operate with an absolutely constant amplitude s_0 and phase ϕ_0. There are always noise sources that make these quantities *fluctuate* (vary around its average value) or *drift* (walk away from its initial value). We will characterize these effects later.

B. Second-Order Linear Oscillators

The simplest model of a mechanical oscillator is a mass on a spring oscillating around a fixed equilibrium value $x_0 = 0$, as shown in Fig. I.1.a. The force that the mass m experiences is given by $F = -kx$, where x is the distance away from the equilibrium value and k is the spring constant. In this case, Newton's force law tells us that $F = ma$, where a is the acceleration or

$$m\frac{d^2x}{dt^2} = -kx. \tag{I.1}$$

Equation I.1 is a second-order linear differential equation with constant coefficients. It will be useful in the future to rewrite this equation as two first-order equations. If we define the velocity $v = dx/dt$, we find that Eq. (I.1) becomes

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{k}{m}x. \tag{I.2}$$
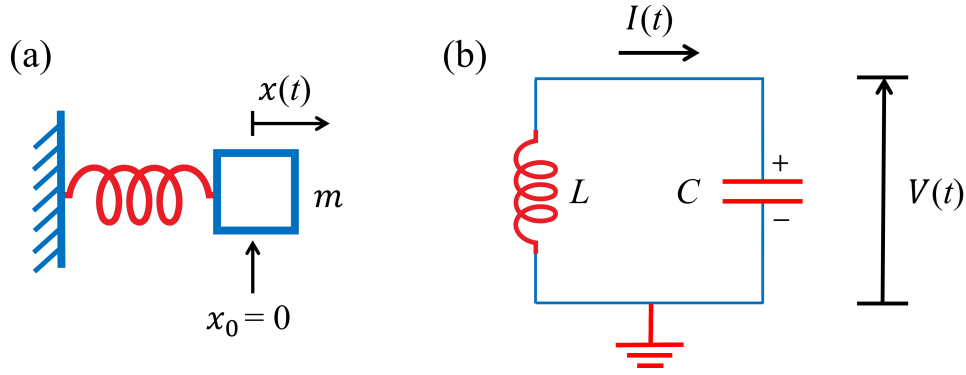
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Figure I.1: Second-order linear oscillators. (a) Mechanical oscillator. A mass on a spring. (b) Electrical oscillator. An *LC* (inductor-capacitor) circuit.

The solution to Eq. (I.2) can be written in a number of different ways, each of which has its uses. One way to write the solution is

$$x(t) = x_0 \cos(\omega t + \phi_0), \quad v(t) = -\omega x_0 \sin(\omega t + \phi_0), \tag{I.3}$$

where $\omega = (k/m)^{1/2}$. The constants x_0 and ϕ_0 are determined by the *initial conditions* of the oscillator, or, in other words, the state that the oscillator is in at the point in time that we pick to be the origin, $t = 0$. We see that $x_i \equiv x(t = 0) = x_0 \cos \phi$ and $v_i \equiv v(t = 0) = -\omega x_0 \sin \phi$, so that $x_0 = (x_i^2 + v_i^2/\omega^2)^{1/2}$ and $\phi_0 = \tan^{-1}(-v_i/\omega x_i)$. Another way to write the solution is

$$x(t) = x_c \cos \omega t - x_s \sin \omega t, \quad v(t) = -\omega x_c \sin \omega t - \omega x_s \cos \omega t, \tag{I.4}$$

where $x_c = a \cos \phi$ and $x_s = a \sin \phi$. A third way to write to write this solution is

$$x(t) = \frac{1}{2}\tilde{x}\exp(i\omega t) + \frac{1}{2}\tilde{x}^* \exp(-i\omega t), \quad v(t) = \frac{i}{2}\omega\tilde{x}\exp(i\omega t) - \frac{i}{2}\omega\tilde{x}^* \exp(-i\omega t). \tag{I.5}$$

where $i \equiv \sqrt{-1}$. The constant $\tilde{x}$ is a complex number, so that $\tilde{x} = \tilde{x}_1 + i\tilde{x}_2$, where both $\tilde{x}_1$ and $\tilde{x}_2$ are real. We have $\tilde{x}^* = \tilde{x}_1 - i\tilde{x}_2$ is the complex conjugate of $\tilde{x}$, and we note that $\exp(\pm i\omega t) = \cos \omega t \pm i \sin \omega t$. It follows that $\tilde{x}_1 = x_c$ and $\tilde{x}_2 = x_s$, so that $\tilde{x} = x_c + ix_s = x_0 \exp(i\phi_0)$. It might seem as though the use of complex numbers unnecessarily complicates things, but, in fact, their use greatly simplifies the mathematical discussions. The phase of the complex number $\tilde{x}$ corresponds to the phase offset in the real displacement $x(t)$.

A point on notation: Engineers typically use $j = \sqrt{-1}$, instead of i. Mathematicians and physicists typically use i. In optical engineering, both notations can be found.

The simplest model of an electrical oscillator is an *LC* circuit, shown in Fig. I.1.b. In this case, we find that

$$\frac{dV(t)}{dt} = \frac{1}{C}I(t), \quad \frac{dI(t)}{dt} = -\frac{1}{L}V(t), \tag{I.6}$$

where L is the inductance and C is the capacitance. This equation is once again a linear second order differential equation. Mathematically, it is identical to the mechanical oscillator, even though

the physical system is completely different. Same equations; same solutions! So, we can just write down the solution,

$$V(t) = \frac{1}{2}\tilde{V}\exp(i\omega t) + \text{c.c.}, \quad I(t) = \frac{i}{2Z_0}\tilde{V}\exp(i\omega t) + \text{c.c}, \tag{I.7}$$

where c.c. is short for complex conjugate, $\omega = 1/(LC)^{1/2}$, $Z_0 = (L/C)^{1/2}$ is the characteristic impedance of the circuit, and $\tilde{V} = \tilde{V}_1 + i\tilde{V}_2$ is determined by the initial conditions. We have $\tilde{V}_1 = V(t=0)$ and $\tilde{V}_2 = -I(t=0)Z_0$.

C. Visualizing the Solutions

In order to understand these solutions, it is useful to plot them. We will use `MATLAB` for this purpose. `MATLAB` is a high-level programming language that has embedded in it many special functions and routines that are necessary for engineering and physics. It also has numerous routines for plotting. UMBC has a license; so, it is available to all UMBC students for free.

When plotting solutions, it is necessary to pick a *normalization* for the quantities since computers work with non-dimensional quantities. It is possible to normalize with respect to the standard SI (Système International) units. The SI units are the basic units in terms of which all other units are defined [s = second, kg = kilogram, m = meter, A = ampere, K = kelvin, mol = mole, cd = candela]. However, it is usually better to work in units that correspond to the physical system. In normalizing time, we are typically interested in times that are short compared to a second; so, we might use units of ms (milliseconds, 10^{-3} s), μs (microseconds, 10^{-6} s), ns (nanoseconds, 10^{-9} s), or ps (picoseconds, 10^{-12} s). In normalizing frequencies, we are typically interested in frequencies that are bigger than 1 Hz; so, we might use units of kHz (kilohertz, 10^3 Hz), MHz (megahertz, 10^6 Hz), GHz (gigahertz, 10^9 Hz), or THz (terahertz, 10^{12} Hz).

In a typical laboratory experiment with a mass on a spring, the mass might equal 10 grams or 10^{-2} kg in standard SI units. The spring might have a spring constant of 10 N/m. In this case, the radial frequency is given by $\omega = [10/10^{-2}]^{1/2} = 31.3$ s^{-1}, so the usual frequency is given by $f = \omega/2\pi = 5.03$ Hz, which means that the weight will oscillate about five times per second. Hence, normalizing with respect to the second makes sense. A typical maximum length x_0 might be on the order of 1 cm (10^{-2} m), so that normalizing lengths with respect to 1 cm makes sense. In this case, we find that the maximum velocity is given by $\omega x_0 = 0.316$ m/s.

If we consider a typical laboratory experiment with an *LC* circuit, the capacitor might have a capacitance of 100 pF (10^{-10} F = 10^{-10} s^{4}A^{2}kg^{-1}m^{-2}) and the inductor might have an inductance of 1 mH (10^{-3} H = 10^{-3} s^{-2}A^{-2}kg m^2). In this case, we find that $\omega = 3.16 \times 10^6$ s^{-1}, so that $f = 503$ kHz or $f = 0.503$ MHz. So, it makes sense to work in units of kHz or perhaps MHz. In 10 μs, there are about five oscillations. If the amplitude of the voltage is 1 V, then the characteristic impedance is given by $Z_0 = (10^{-3}/10^{-10})^{1/2}$ ohms $= 3.16 \times 10^3$ Ω = 3.16 kΩ. The amplitude of the current is then given by $I_0 = 0.316$ mA or 316 μA.

Figure I.2.a shows plots of the voltage and current as a function of time for five periods. Figure I.2.b shows phase plots in which both the current and voltage are plotted as time varies. Phase plots are useful as they show qualitative features of the evolution that time plots don't reveal, and we will use them many times. We see that the phase plots close in on themselves, which corresponds to the pattern repeating periodically, as is required for a good oscillator. Here, we show three different cases, corresponding to $V_0 = 0.5$, 0.7, and 0.9. The curves are elliptical in shape
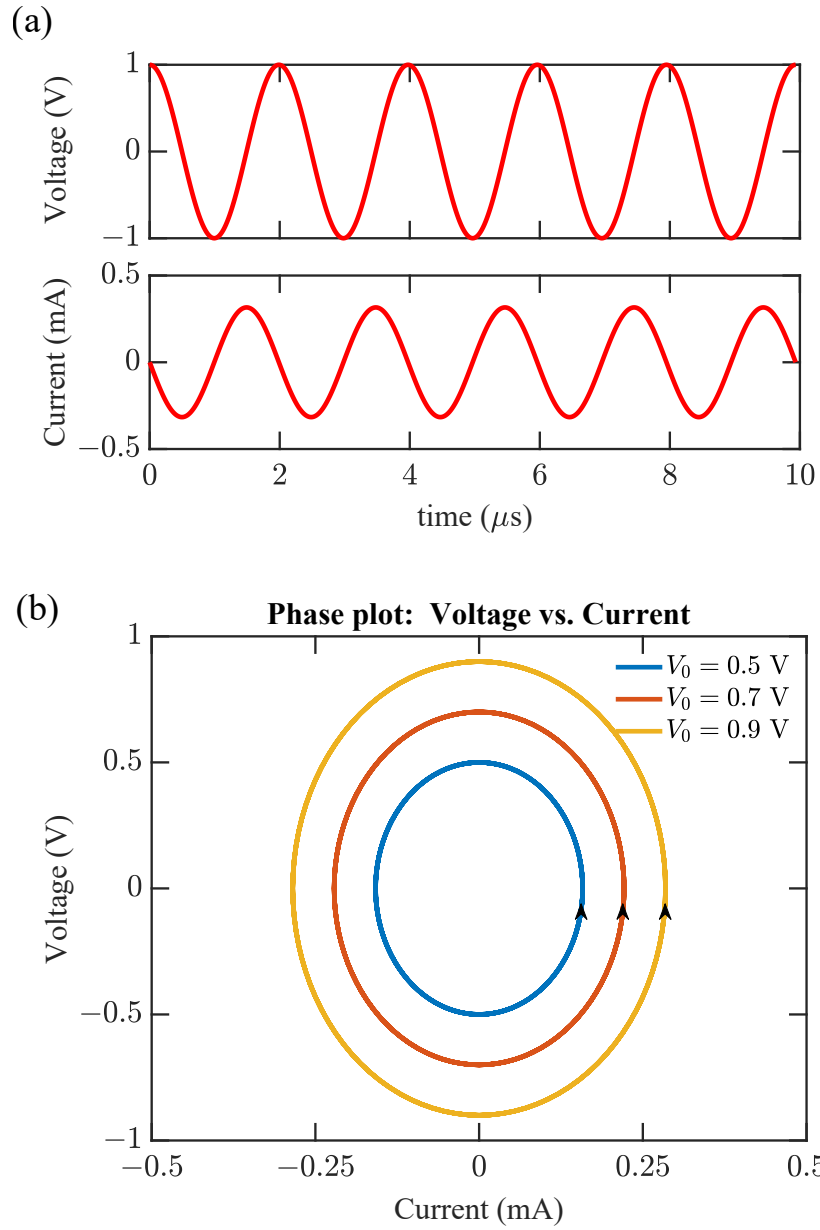
Figure I.2: Plots of the voltage and current (a) as a function of time and (b) as phase plots.

because the oscillator energy is constant. The energy of the mechanical oscillator is given by

$$U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \tag{I.8}$$

and the energy of the electrical oscillator is given by

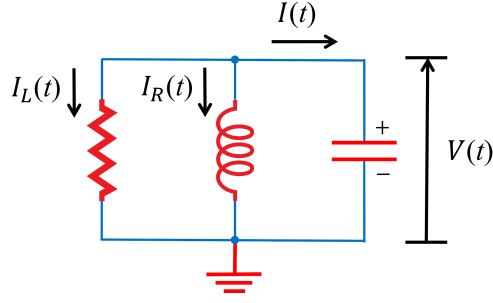$$U = \frac{1}{2}LI^2 + \frac{1}{2C}V^2. \tag{I.9}$$

Figure I.3: Parallel *RLC* circuit.

D.   Damped-Driven Oscillators

In reality, all physical systems are damped. In the case of the mechanical oscillators, our equations become

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\alpha v - \frac{k}{m}x, \tag{I.10}$$

For the electrical oscillator, if we consider the *RLC* circuit that we show in Fig. I.3, we find

$$\frac{dV}{dt} = \frac{1}{C}I, \quad \frac{dI}{dt} = -\frac{1}{RC}I - \frac{1}{L}V, \tag{I.11}$$

where R is the resistance. Loss in this circuit is reduced when the resistance is large so that there is little current flow through the resistor. Focusing on the electrical oscillator, we find that Eq. (I.11) has the general solution

$$V(t) = \frac{1}{2}\tilde{V}_+ \exp(\lambda_+ t) + \frac{1}{2}\tilde{V}_- \exp(\lambda_- t), \quad I(t) = \frac{\lambda_+}{2}\tilde{V}_+ \exp(\lambda_+ t) + \frac{\lambda_-}{2}\tilde{V}_- \exp(\lambda_- t), \tag{I.12}$$

where

$$\lambda_\pm = -\frac{1}{2RC} \pm \left[\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}\right]^{1/2}. \tag{I.13}$$

We see that if $(1/2RC)^2 > 1/LC$, then both λ_+ and λ_- are real, while if $1/LC > (1/2RC)^2$, then λ_+ and λ_- are complex conjugate numbers. For oscillators, we are interested in the second case, where the damping is small. In this case, we find $\lambda_+ = i\omega - \gamma$, where

$$\omega = \left[\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2\right]^{1/2} = \frac{1}{(LC)^{1/2}}\left[1 - \frac{1}{8}\frac{L}{R^2 C} + \cdots\right]^{1/2} \simeq \frac{1}{(LC)^{1/2}}, \tag{I.14}$$

where we have written a Taylor expansion for ω, and we have just kept the lowest-order term. We find $\gamma = 1/2RC$. In this limit, we may write

$$V(t) = \frac{1}{2}\tilde{V}\exp[(i\omega - \gamma)t] + \text{c.c.}, \quad I(t) = \frac{i}{2Z}\tilde{V}\exp[(i\omega - \gamma)t] + \text{c.c.}, \tag{I.15}$$

where $1/Z = (\omega + i\gamma)C$. Hence, the voltage and current are no longer exactly $\pi/2$ out of phase. Writing the voltage and current explicitly as real quantities, we find
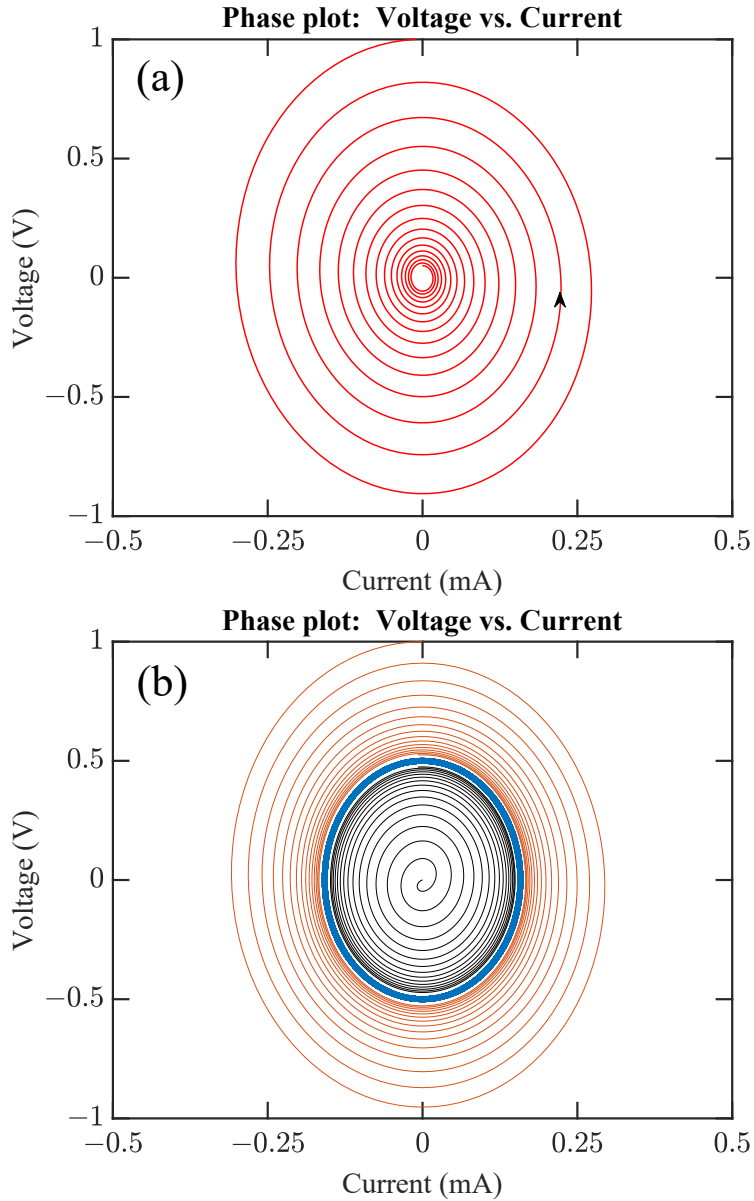
Figure I.4: Phase plots of driven-damped oscillators. (a) The damped oscillator without driving. The phase trajectory tends to zero. (b) The driven oscillator with a single driving frequency and two different initial conditions. The trajectory in both cases tends toward the driven solution, shown in blue.

$$V(t) = V_0 \cos(\omega t + \phi_0) \exp(-\gamma t),$$
$$I(t) = -\frac{V_0}{Z_0} \sin(\omega t + \phi_0 + \phi_{\mathrm{off}}) \exp(-\gamma t),$$
(I.16)

where V_0 and ϕ_0 are two constants determined by the initial conditions, and $\phi_{\mathrm{off}} = \tan^{-1}(\gamma/\omega)$

Returning to the oscillator circuit that we considered in Sec. I.C and setting the resistance R equal to 50 kΩ, $V_0 = 1$ V, and $\phi_0 = 0$, we show the evolution in Fig. I.4.a. The resistance causes the voltage and current to spiral into the origin, ultimately damping away completely.
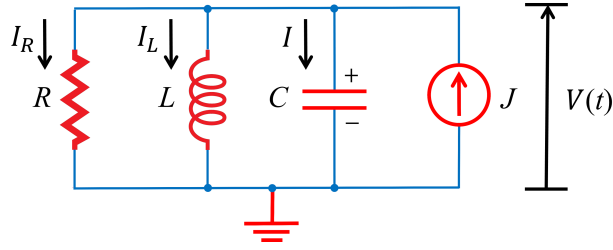
Figure I.5: *RLC* circuit with a current source.

D.1. *A single frequency driver*

This "death spiral" is of course unacceptable in real oscillators, and it must be compensated by a source of energy that makes up for the loss. So, for example, in the *RLC* circuit that we are considering, we can add a current source, as shown in Fig. I.5. Writing the time derivative of the current source $J(t)$ as $J'(t)$, we find that the equations that govern the voltage and current become

$$\frac{dV}{dt} - \frac{1}{C}I = 0, \quad \frac{dI}{dt} + \frac{1}{RC}I + \frac{1}{L}V = J'. \tag{I.17}$$

Equation (I.17) is referred to as an *inhomegeneous* linear equation because of the driving term, as opposed to Eqs. (I.1), (I.2), (I.6), (I.10), and (I.11), all of which are *homogeneous* linear equations. The term J' is referred to as the *inhomogeneous term*.

We begin by considering the case where the driver is a pure oscillating signal, so that

$$J = J_0 \cos(\omega t + \phi_0) = \frac{1}{2}\tilde{J}\exp(i\omega t) + \text{c.c.}, \tag{I.18}$$

where J_0 and ϕ_0 are constants, and $\tilde{J} = J_0 \exp(i\phi_0)$ is a complex constant. We will denote the resonant frequency of the oscillator as $\omega_{\mathrm{r}} = (1/LC)^{1/2}$. The driven solution is referred to as a *particular solution*.

It is useful to choose the time origin so that $\phi_0 = 0$, in which case $\tilde{J} = J_0$ is purely real. Because our equations are linear, we can find the solutions for $V(t)$ and $I(t)$ by just setting $J(t) = \tilde{J}\exp(i\omega t)$, so that $J' = i\omega\tilde{J}\exp(i\omega t)$ and then taking the real part at the end. That turns out to be the simplest approach mathematically. There will be an initial transient that damps out, after which both $I(t)$ and $V(t)$ will be proportional to $\exp(i\omega t)$. Writing $I(t) = \tilde{I}\exp(i\omega t)$ and $V(t) = \tilde{V}\exp(i\omega t)$, we find

$$\tilde{V} = \frac{i\omega}{C}\frac{\tilde{J}}{\omega_{\mathrm{r}}^2 - \omega^2 + 2i\omega\gamma} = \frac{i\omega}{C}\frac{\omega_{\mathrm{r}}^2 - \omega^2 - 2i\omega\gamma}{(\omega_{\mathrm{r}}^2 - \omega^2)^2 + 4\omega^2\gamma^2}\tilde{J},$$

$$\tilde{I} = -\frac{\omega^2\tilde{J}}{\omega_{\mathrm{r}}^2 - \omega^2 + 2i\omega\gamma} = -\frac{\omega^2(\omega_{\mathrm{r}}^2 - \omega^2 - 2i\omega\gamma)}{(\omega_{\mathrm{r}}^2 - \omega^2)^2 + 4\omega^2\gamma^2}\tilde{J}. \tag{I.19}$$

We then find that the real driven solutions, $V_D(t)$ and $I_D(t)$ are given by

$$V_D(t) = \frac{\omega}{\omega_r} Z_0 J_0 A_D \cos(\omega t + \phi_D) = \frac{\omega}{\omega_r} Z_0 J_0 A_D \left[\cos\phi_D \cos\omega t - \sin\phi_D \sin\omega t\right]$$
$$= V_{Dc}\cos\omega t - V_{Ds}\sin\omega t,$$

$$I_D(t) = -\frac{\omega^2}{\omega_r^2} J_0 A_D \sin(\omega t + \phi_D) = -\frac{\omega^2}{\omega_r^2} J_0 A_D \left[\sin\phi_D \cos\omega t + \cos\phi_D \sin\omega t\right],$$
$$-\frac{\omega}{\omega_r}\frac{1}{Z_0}\left(V_{Ds}\cos\omega t + V_{Dc}\sin\omega t\right),$$

(I.20)

$$A_D = \frac{\omega_r^2}{[(\omega_r^2 - \omega^2)^2 + 4\omega^2\gamma^2]^{1/2}},$$

$$\sin\phi_D = \frac{\omega_r^2 - \omega^2}{[(\omega_r^2 - \omega^2)^2 + 4\omega^2\gamma^2]^{1/2}}, \quad \cos\phi_D = \frac{2\omega\gamma}{[(\omega_r^2 - \omega^2)^2 + 4\omega^2\gamma^2]^{1/2}},$$

(I.21)

$$V_{Dc} = A_D\cos\phi_D, \quad V_{Ds} = A_D\sin\phi_D.$$

We now rewrite the transient solution in Eq. (I.16) in the form

$$V_T(t) = V_0\cos(\omega_{osc}t + \phi_0)\exp(-\gamma t) = [V_c\cos\omega_{osc}t - V_s\sin\omega_{osc}t]\exp(-\gamma t),$$

$$I_T(t) = -\frac{\omega_{osc}}{\omega_r}\frac{1}{Z_0}\left[\left(V_c - \frac{\gamma}{\omega_{osc}}V_s\right)\sin\omega_{osc}t + \left(V_s + \frac{\gamma}{\omega_{osc}}V_c\right)\cos\omega_{osc}t\right]\exp(-\gamma t),$$

(I.22)

where $\omega_{osc} = (\omega_r^2 - \gamma^2)^{1/2}$ is the natural frequency of the oscillator, $V_c = V_0\cos\phi_0$, $V_s = V_0\sin\phi_0$, and we note that $\sin\phi_{off} = \gamma/\omega_r$, $\cos\phi_{off} = \omega_{osc}/\omega_r$. Rewriting Eq. (I.16), makes it easy to express the total current in terms of the initial conditions. The total voltage and current are given by $V(t) = V_D(t) + V_T(t)$ and $I(t) = I_D(t) + I_T(t)$. Writing V_c and V_s in terms of the initial conditions, we find

$$V_c = V(t=0) - V_{Dc}, \quad V_s = -\frac{\omega_r}{\omega_{osc}}[Z_0 I(t=0) - V_{Ds}] - \frac{\gamma}{\omega_{osc}}V_c.$$

(I.23)

In Fig. I.4.b, we show the transient evolution of the the voltage $V(t)$ vs. the current $I(t)$ when the driving current $J_0 = 10\ \mu A$, and we set $\omega = \omega_r$. We show two cases. In both cases, the initial current is 0; in the first case, the initial voltage is 0 V, and in the second case, the initial voltage is 1 V. While in both cases, the transient evolution ultimately approaches the driven oscillation, the approach is slow once the trajectory nears the final trajectory, shown in blue. This slow approach is not desirable in practice, and we will show later that it is possible to use nonlinearity to force a faster approach to the final oscillating state.

A point that may appear surprising at first is that the current oscillations are much larger than the driving current. However, most of the current oscillates back and forth between the inductor and the capacitor; a relatively small amount of drive current is needed to compensate for the loss in the resistor. The ratio of the ω_r/γ in our illustration ($\simeq 30$) is relatively small, and our oscillator is in fact a poor oscillator! We chose this small value in order to be able to illustrate the damping and transient effects. In fact, any useful oscillator will have ratios that are many orders of magnitude higher. Some loss is however essential. In order to use the oscillator, we must be able to measure the voltage, and that is done by having a load, which implies resistance and some loss. The same holds true for mechanical oscillators, where the displacement must be measured, or atomic oscillators,

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where emission from a higher energy state to a lower energy state is accompanied by radiation and energy loss. In the case of our electrical oscillator, the energy dissipation U_{diss} during one oscillation period, once the transient oscillations have ended, is given by

$$U_{\text{diss}} = \int_0^{2\pi/\omega} [V^2(t)/R]\, dt = \frac{\pi}{\omega R} Z_0^2 J_0^2 \frac{\omega^2}{\omega_{\text{r}}^2} A_{\text{D}}^2. \tag{I.24}$$

The rate at which energy transfers to the load on average is

$$\frac{d\bar{U}}{dt} = \frac{1}{2R} Z_0^2 J_0^2 \frac{\omega^2}{\omega_{\text{r}}^2} A_{\text{D}}^2. \tag{I.25}$$

D.2. A broadband driver

In the solution that we just obtained, we assumed that the driver is itself a perfect oscillator with an ideal sinusoidal variation in time. However, the purpose of an oscillator is to take an imperfect driver that has many different driving frequencies and turn it into a nearly pure signal. If we now consider a driver that consists of N different driving frequencies, we have

$$J(t) = \sum_{n=1}^N J_n \cos(\omega_n t + \phi_n). \tag{I.26}$$

If we have a broadband source, then we may assume that $J_n = J_0$ is constant over the range of frequencies that can couple efficiently to the oscillator. We then find that

$$\frac{d\bar{U}}{dt} = \frac{1}{2R} Z_0^2 J_0^2 \sum_{n=1}^N \frac{\omega^2}{\omega_{\text{r}}^2} A_{\text{D}}^2(\omega_n). \tag{I.27}$$

In the limit of practical interest, in which $\omega_{\text{r}}/\gamma \gg 1$, we can neglect the difference between ω and ω_{r} in Eq. (I.27), except for the term in the denominator in which $(\omega_{\text{r}}^2 - \omega^2)$ appears. Even this term can be rewritten as $(\omega_{\text{r}}^2 - \omega^2) \simeq 2\omega_{\text{r}}(\omega_{\text{r}} - \omega)$. We now find that

$$A_{\text{D}}^2(\omega) \simeq \frac{1}{4} \frac{\omega_{\text{r}}^2}{(\omega - \omega_{\text{r}})^2 + \gamma^2}. \tag{I.28}$$

We see that the A_{D}^2 falls to half its maximum value when $\Delta\omega \equiv \omega - \omega_{\text{r}} = \gamma$.

In Fig. I.6, we show $a^2 = 4A_{\text{D}}^2\gamma^2/\omega_{\text{r}}^2$ vs. $x = (\omega - \omega_{\text{r}})/\gamma$ in both a linear and a log-log plot. We see that both a^2 and x are *normalized*. They are unitless, so that we obtain a figure that is the same for *any* oscillator, independent of the values of ω_{r} and γ. In Fig. I.6.a, we see that the function tends to zero as $x \to \pm\infty$, which implies that the oscillator is doing what it should; it is filtering the broadband driver. In Fig. I.6.b, we see that the curve has a slope of 0 when $0 < x < 1$ and a slope of -2 when $1 < x < \infty$. The slope of -2 is characteristic of a polynomial falloff proportional to $|\omega - \omega_{\text{r}}|^{-2}$ and is something that can be looked for experimentally. With real oscillators, there will be a noise floor that cuts off the continued falloff $\propto |\omega - \omega_{\text{r}}|^{-2}$.

It is usual to characterize oscillators using their *linewidths* or *quality factors*. The linewidth is the bandwidth between the two points where the response of the oscillator falls to half its maximum value, referred to as the *full width at half maximum* (FWHM) value. In the case that we are considering, we find that the FWHM linewidth is given by γ/π since it is usual to use the standard
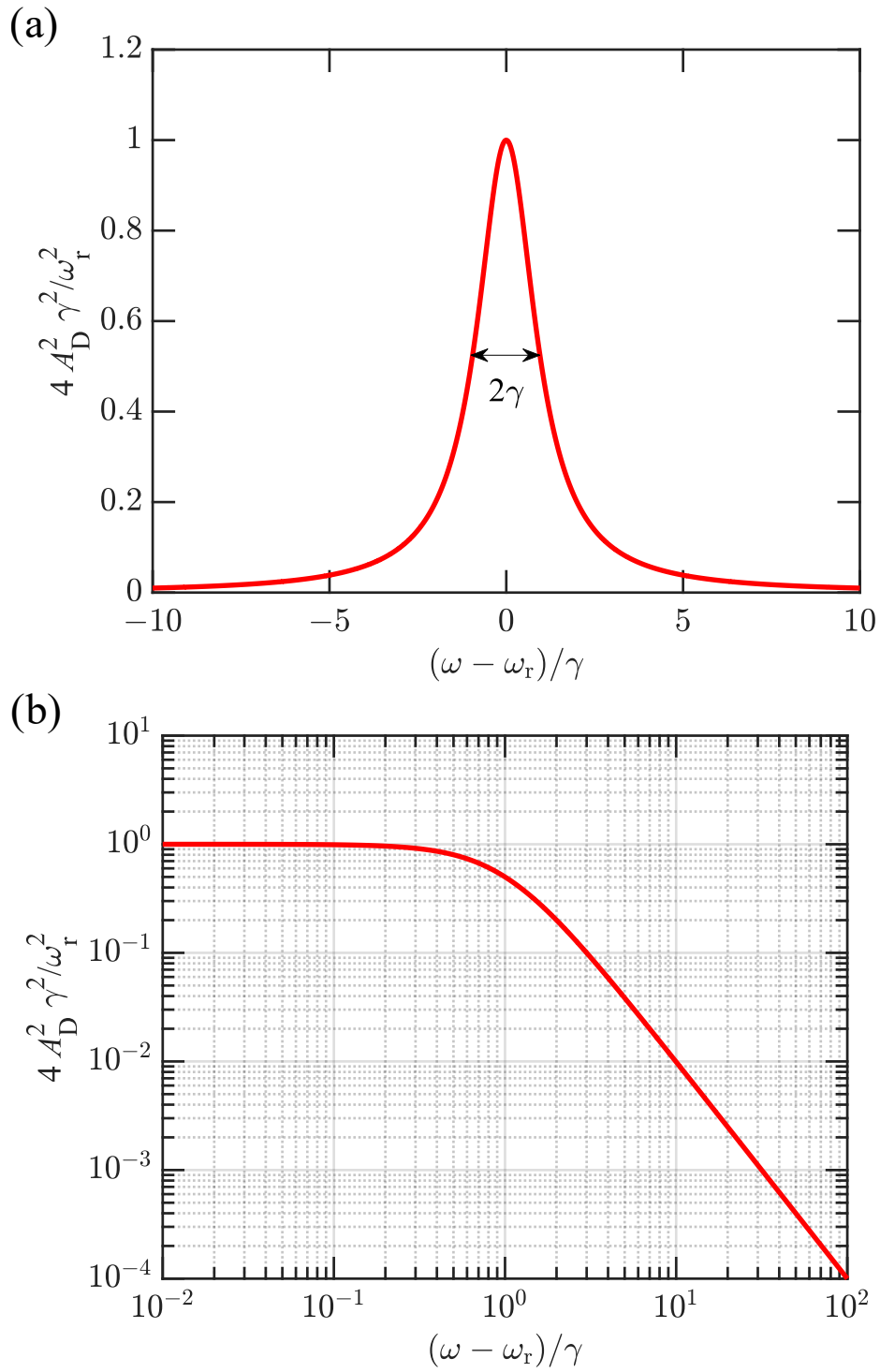
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Figure I.6: (a) Linear and (b) log-log plots of $4A_D^2\gamma^2/\omega_r^2$ vs. $(\omega-\omega_r)/\gamma$. In (a) we illustrate the FWHM width in the angular frequency. In (b) we show the characteristic slopes of $4A_D^2\gamma^2/\omega_r^2$ when $0 \le x < 1$ and -2 when $x > 1$ in a log-log plot.

frequency, rather than the angular frequency when quoting linewidths. The quality factor (Q) is de-

fined in two different ways that are not exactly equivalent. The first is $Q = f/\Delta f$, where Δf is the FWHM linewidth and f is the resonant frequency. For the linear oscillator, we find that $Q = \omega_{\mathrm{r}}/2\gamma$. Another way to define Q is

$$Q = 2\pi \times \frac{\text{energy stored}}{\text{energy dissipated per cycle}}. \tag{I.29}$$

For the electrical oscillator that we considered, the energy stored is given by $U_{\mathrm{stored}} = (L/2)I_{\mathrm{max}}^2 = (1/2C)V_{\mathrm{max}}^2$, where I_{max} is the maximum current flowing through the inductor, V_{max} is the maximum voltage at the capacitor, and we note that the stored energy oscillates back and forth between the two. From Eq. (I.20), we see that $(L/2)I_{\mathrm{max}}^2 = (L/2)J_0^2 A_{\mathrm{D}}^2$. From Eq. (I.24), we see that in the limit of interest to us $(\omega \simeq \omega_{\mathrm{r}})$, $U_{\mathrm{diss}} = (2\pi/\omega_{\mathrm{r}})(L/2RC)J_0^2 A_{\mathrm{D}}^2$. Recalling that $\gamma = 1/2RC$, we find that $Q = 2\pi(U_{\mathrm{stored}}/U_{\mathrm{diss}}) = 2\gamma/\omega_{\mathrm{r}}$, which is the same result that we obtained with our previous definition. These two definitions are equivalent for any linear oscillator with a single oscillation frequency, regardless of whether the oscillator is electrical or mechanical.

E. Matrix Representations: Eigenvalues and Eigenvectors

To find the oscillation frequencies in Eq. (I.3), (I.7), and (I.14), we can use a simple algebraic approach. To find $\tilde{V}$ and $\tilde{I}$ in Eq. (I.19), we can also use a simple algebraic approach in which we first eliminate one of the variables, solve for the second, and substitute to find the first. However, this *ad hoc* approach no longer works well when we consider more complex system. Instead, it is better to use the methods of linear algebra in which we use matrix representations.

We begin by defining a 2×1 column vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{I.30}$$

where $u_1 = x$ and $u_2 = v$ for the mechanical oscillator, and $u_1 = V$ and $u_2 = I$ for the electrical oscillator. We can now write the equations that govern the oscillator as

$$\frac{d\mathbf{u}}{dt} = \mathsf{A}\mathbf{u}, \tag{I.31}$$

where

$$\mathsf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{I.32}$$

is a 2×2 matrix. The rule for multiplying a 2×2 matrix and 2×1 column vector is

$$\mathbf{v} = \mathsf{A}\mathbf{u} \leftrightarrow v_m = \sum_{n=1}^{2} A_{mn}u_n, \tag{I.33}$$

where $\mathbf{v}$ is also a 2×1 column vector. For the mechanical driven-damped oscillator, we have $A_{11} = 0$, $A_{12} = 1$, $A_{21} = -k/m$, and $A_{22} = -\alpha$. For the electrical oscillator, we have $A_{11} = 0$, $A_{12} = 1/C$, $A_{21} = -1/L$, $A_{22} = -1/RC$.

To find the oscillation frequencies, we search for solutions that have the form $d\mathbf{u}/dt = \lambda\mathbf{u}$ since we know that the solution of (almost) any linear ordinary differential equation is composed of the sum of functions that vary exponentially in time. (The small caveat is that in some cases,
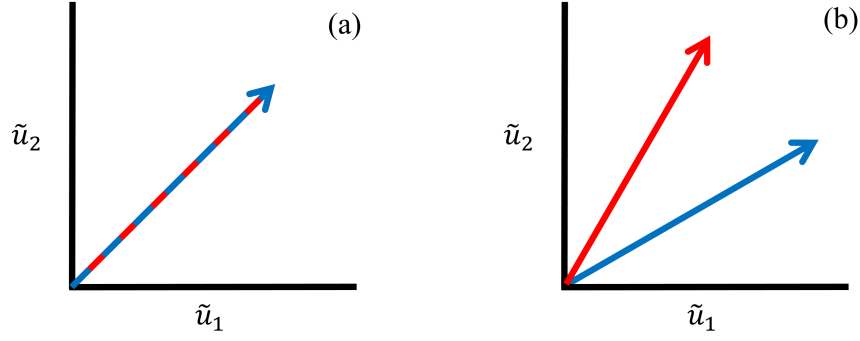
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Figure I.7: Schematic illustration of (a) non-independent and (b) independent eigenvalue equations. When the equations are not independent, they define lines that lie on top of each other, shown as a red-blue dashed line. When the equations are independent, they define lines that only meet at $\tilde{u}_1 = \tilde{u}_2 = 0$.

the exponentials have to multiplied by polynomials in time.) Writing $\mathbf{u} = \tilde{\mathbf{u}} \exp(\lambda t)$, Eq. (I.31) becomes

$$\lambda \tilde{\mathbf{u}} = A \tilde{\mathbf{u}} \quad \text{or} \quad (A - \lambda I)\tilde{\mathbf{u}} = 0, \tag{I.34}$$

where I is the identity matrix, whose elements are given by $I_{11} = 1$, $I_{12} = 0$, $I_{21} = 0$, and $I_{22} = 1$. This equation is an *eigenvalue* equation. Eigenvalue equations play an important role in almost every area of science and engineering; so, there are many textbooks that describe their properties, as well as computational implementations that solve these equations. We will take advantage of the MATLAB implementations. We see that one solution to Eq. (I.34) is the trivial solution $\tilde{\mathbf{u}} = 0$. In order for there to be non-trivial solutions, the two equations that make up the matrix equation $(A_{11} - \lambda)\tilde{u}_1 + A_{12}\tilde{u}_2 = 0$ and $A_{12}\tilde{u}_1 + (A_{22} - \lambda)\tilde{u}_2 = 0$ must be effectively the same equation. If we plot $\tilde{u}_2$ vs. $\tilde{u}_1$, we see that both equations correspond to lines that pass through zero. Either the slopes on the same, in which case they corresponds to the same line, as shown in Fig. I.7.a, or the slopes are different, shown in Fig. I.7.b, in which case the only solution is $\tilde{u}_1 = \tilde{u}_2 = 0$. In the former case, we say that the equations are *not independent*. More generally, we can also say that the determinant of the matrix $A - \lambda I = 0$, which is written $\det(A - \lambda I) = 0$. For large dimensional systems, the eigenvalues and the oscillating frequencies are still determined by the condition $\det(A - \lambda I) = 0$. We will give a general definition of the determinant later. The determinant of any 2×2 matrix B is given by $B_{11}B_{22} - B_{12}B_{21}$ and for our eigenvalue equation that implies $(A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21} = 0$ or $\lambda^2 - \lambda(A_{11} + A_{22}) + (A_{11}A_{22} - A_{12}A_{21}) = 0$. We thus obtain a quadratic equation for λ. Substituting in the expressions for either the mechanical or electrical oscillator, we can obtain the same expressions for the complex oscillation frequencies that we obtained before. More generally, these values of λ are referred to as *eigenvalues*

Once we have found the eigenvalues, we can find corresponding values of $\tilde{\mathbf{u}}$ that solve these equations. These are referred to as *eigenvectors*. For each eigenvalue, we can use either of the two rows of the eigenvalue equation. If we consider for example the electrical oscillator without damping, we find that eigenvectors are given by

$$\mathbf{e}_+ = \begin{bmatrix} 1 \\ i/Z_0 \end{bmatrix}, \qquad \mathbf{e}_- = \begin{bmatrix} 1 \\ -i/Z_0 \end{bmatrix}, \tag{I.35}$$

corresponding respectively to the eigenvalues $\lambda_+ = i\omega_r$ and $\lambda_- = -i\omega_r$. Since the equations are linear, any eigenvector when multiplied by a constant is also an eigenvector. These correspond

to all the points on the line that we show schematically in Fig. I.8.a. So, the general solution of Eq. (I.31) becomes

$$\mathbf{u} = \frac{1}{2}\tilde{u}_+\mathbf{e}_+ \exp(i\omega_r t) + \frac{1}{2}\tilde{u}_-\mathbf{e}_- \exp(-i\omega_r t), \tag{I.36}$$

where $\tilde{u}_+$ and $\tilde{u}_-$ are complex constants that depend on the initial conditions. At $t = 0$, we see that

$$\mathbf{u}(t=0) = \begin{bmatrix} V(t=0) \\ I(t=0) \end{bmatrix} = \begin{bmatrix} (1/2)(\tilde{u}_+ + \tilde{u}_-) \\ (i/2Z_0)(\tilde{u}_+ - \tilde{u}_-) \end{bmatrix}, \tag{I.37}$$

from which we conclude $\tilde{u}_+ = V(t=0) - iZ_0I(t=0)$ and $\tilde{u}_- = V(t=0) + iZ_0I(t=0)$. We see that $\tilde{u}_+$ and $\tilde{u}_-$ are complex conjugates, which ensures $\mathbf{u}$ is real at all times. Comparing Eq. (I.37) to Eq. (I.7), we see that $\tilde{V} = \tilde{u}_+$. This solution is consistent with our earlier results. So, we have learned nothing new. However, this approach is the best approach to use when considering higher-dimensional oscillators.

To solve the driven-damped oscillator equations use matrix methods, we first rewrite Eq. (I.17) in the form

$$\frac{d\mathbf{u}}{dt} - A\mathbf{u} = \mathbf{D}, \tag{I.38}$$

where

$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & 1/RC \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 \\ J' \end{bmatrix}. \tag{I.39}$$

We assume that our driver $\mathbf{D}$ is at a single frequency, so that $\mathbf{D} = \tilde{\mathbf{D}}\exp(i\omega t)$, where $\tilde{\mathbf{D}}$ is a known constant vector, and we search for solutions in the form $\mathbf{u} = \tilde{\mathbf{u}}\exp(i\omega t)$, where $\tilde{\mathbf{u}}$ is an unknown constant vector. Writing out these vectors, we have

$$\tilde{\mathbf{D}} = \begin{bmatrix} 0 \\ i\omega\tilde{J} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{u}} = \begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix}. \tag{I.40}$$

Equation (I.38) now becomes

$$B\tilde{\mathbf{u}} = \tilde{\mathbf{D}}, \tag{I.41}$$

where $B = i\omega I - A$. To solve Eq. (I.41), we may use the inverse of B, written as B^{-1}. This matrix satisfies the relation $B^{-1}B = BB^{-1} = I$. It then follows that

$$\tilde{\mathbf{u}} = B^{-1}\tilde{\mathbf{D}}. \tag{I.42}$$

In general, it is not efficient to solve matrix equations using the inverse when the system of equations becomes large. It is better to use *Gauss-Jordan elimination*. However, this approach works well for the second-order system that we are considering here. The inverse of a matrix B is given by

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} B_{22} & -B_{12} \\ -B_{12} & B_{11} \end{bmatrix}, \tag{I.43}$$

which becomes explicitly in this case

$$B^{-1} = \frac{1}{[i\omega(i\omega - 1/RC) + 1/LC]} \begin{bmatrix} i\omega + 1/RC & 1/C \\ -1/L & i\omega \end{bmatrix}. \tag{I.44}$$

Operating with B^{-1} on $\tilde{\mathbf{D}}$ and using the relations $1/RC = 2\gamma$, $1/LC = \omega_r^2$, we obtain Eq. (I.19).

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Figure I.8: MATLAB output for the cases considered in Fig. I.4. (a) The eigenvalues and eigenvectors for the damped oscillations, shown in Fig. I.4.a. (b) The driven voltage and current solutions shown in Fig. I.4.b.

Due to the importance of linear algebra in a wide variety of applications, MATLAB has implemented a powerful set of algorithms to solve eigenvalue problems and linear systems of equations. For the numerical examples that we presented in Fig. I.4, we have $C = 10^{-10}$, $L = 10^{-3}$, and $R = 5 \times 10^4$ in appropriate SI units. Explicitly, we then have

$$A = \begin{bmatrix} 0 & 10^{10} \\ -10^3 & -2 \times 10^5 \omega \end{bmatrix}. \tag{I.45}$$

In general, it is not good computational practice to use numbers with large exponents. It is better to use quantities that are all on the order of one by appropriately renormalizing quantities. If we use microseconds instead of seconds and we use milliamperes instead of amperes, our matrix becomes

$$A = \begin{bmatrix} 0 & 10 \\ -1 & -0.2 \end{bmatrix}. \tag{I.46}$$

and we see that none of the quantities differs too much from one. This sort of dimensional analysis also gives us an immediate feel for the rough value that quantities will have. Frequencies will be on the order a megahertz, voltages will be on the order of one volt, and currents will be on the order of milliamperes. That is consistent with the results that we already plotted in Fig. I.2.

To find the eigenvalues and eigenvectors, we may use MATLAB's eig routine. The command [E,L] = eig(A) produces the eigenvalues $\lambda_\pm = -0.1 \pm 3.1607i$, which are the diagonal elements of the matrix L. We show the output in Fig. I.6.a. This command also produces the eigenvectors $\mathbf{e}_\pm = [0.9535, -.0095 \pm 0.3014]^t$, where t denotes the transpose, so that

$$[0.9535, -.0095 \pm 0.3014]^t = \begin{bmatrix} 0.9535 \\ -.0095 \pm 0.3014 \end{bmatrix}.$$

The eigenvectors appear as the columns of the matrix $\mathsf{E} = [\mathbf{e}_+, \mathbf{e}_-]$. MATLAB uses the l^2 norm for eigenvectors in which the sum of the absolute squares equals 1. In order to put the eigenvectors in the form where the first element of $\mathbf{e}_+$ and $\mathbf{e}_-$ both equal 1.0, we divide E by E_{11}, which yields $\mathbf{e}_\pm = [1.0, -0.01 \pm 0.3161i]$. We infer that $\gamma = 0.1 \; \mu s^{-1}$ and $\omega_{\mathrm{osc}} = 3.1607 \; \mu s^{-1}$, corresponding to an oscillation frequency of 0.503 MHz or 503 kHz, which is what we found earlier. We infer as well that $i/Z = (i/Z_0)(\omega_{\mathrm{osc}}/\omega_{\mathrm{r}} + i\gamma/\omega_{\mathrm{r}}) = -0.01 + 0.3161i \; k\Omega^{-1}$, so that $Z_0 = |Z| = 3.162 \; k\Omega$, which is again consistent with our earlier result.

To find $\tilde{\mathbf{u}} = [\tilde{V}, \tilde{I}]^t$ for the case driven solution that we considered in Fig. I.4, we first recall that $\omega = \omega_{\mathrm{r}} = \sqrt{10} = 3.1623 \; \mu s^{-1}$ in the case that we considered. We also recall that $i\omega\tilde{J} = i\sqrt{10} \times 0.01 \; \mathrm{mA}\text{-}\mu s^{-1}$. Hence, we find

$$B = \begin{bmatrix} i\sqrt{10} & -10 \\ 1 & i\sqrt{10} + 0.2 \end{bmatrix}, \qquad \tilde{\mathbf{D}} = \begin{bmatrix} 0 \\ i0.01\sqrt{10} \end{bmatrix}. \tag{I.47}$$

To find $\tilde{\mathbf{u}} = [\tilde{V}, \tilde{I}]^t$, we use the MATLAB command u = B\D. This command uses Gauss-Jordan elimination to solve for $\tilde{\mathbf{u}}$. We show the results in Fig. I.6.b. We find $\tilde{V} = 0.5$ V and $\tilde{I} = 0.1581i$ mA, which equals 158 μA. These results are consistent with our prior results.

Exercises

These exercises combine investigative exercises that require you to do some on-line research (I), mathematical or computational exercises that require you to verify the steps in the calculations of this section or extend them (M), and experimental exercises that require you to build something to test the ideas in this section (E).

1. (I) The most commonly used oscillators are quartz crystal oscillators. What is their operating principle? What is its operating frequency range? What are typical quality factors?

2. (I) Rubidium and sapphire crystal oscillators are alternatives to quartz crystal oscillators. What are their advantages and disadvantages?

3. (I) The best oscillators are based on atomic transitions. Cesium transitions are the basis of modern-day atomic clocks. There is increasing interest in using transitions of an ytterbium lattice, and future clocks are likely to based on these transitions. What are the operating frequencies and linewidths of these oscillators? What are the corresponding quality factors?

4. (M) For the mechanical oscillator that is described in Sec. I.C, modify the code `OscillatorI2` (part b) to make a phase plot of x vs. v for five cases where the maximum excursion ranges between 1 cm and 4 cm. Calculate the energy that is stored in the oscillator for each of these cases.

5. (M) Verify Eq. (I.5) by substitution into the governing equation, Eq. (1.2), and relate $\tilde{x}$ and $\tilde{v}$ to the initial conditions.

6. (I and M) If we consider hydrogen-iodide (HI), which is a diatomic molecule, the hydrogen atom will vibrate in the potential well of the iodine atom. While a real understanding of the molecule must use quantum mechanics, a basic understanding can be obtained by treating the hydrogen atom as a simple second-order oscillator. The frequency of vibration can be calculated from the wavenumber of the emissions, which is 2230 cm^{-1} and the approximate magnitude of the oscillations can be calculated from the dimension of the molecule, which is 161 pm. The mass of the hydrogen atom is close to 1 atomic unit. Calculate the frequency of vibration of the hydrogen atom and its mass in SI units. Use that information to calculate the effective spring constant. Suppose that the energy stored in the vibrational motion is given by hf, where h is Planck's constant and f is the oscillation frequency, how large an excursion is the hydrogen atom making and how does that compare to the size of the molecule?

7. (M) Derive Eq. (I.11) using Kirchhoff's current law for the circuit of Fig. I.3.

8. (M) Verify Eqs. (I.12–I.16) by substitution into the governing equation, Eq. (I.11).

9. (M) Derive equivalent expressions to Eqs. (I.12-I.16) for the mechanical oscillator, described by Eq. (1.10). For the example parameters of Sec. I.C, what damping rate α would correspond to $\gamma/\omega_r = 1/100$, $1/10\sqrt{10}$, and $1/10$? Modify `OscillatorI4` (part a) to make phase plots for each of these cases.

10. (M) Verify Eqs. (1.19–1.23) by substitution into the governing equation, Eq. (I.17).

11. (M) Modify `OscillatorI4` (part b) to plot the driven oscillation for five driving frequencies, equal to $0.25\omega_r$, $0.5\omega_r$, $1.0\omega_r$, $1.5\omega_r$, and $2.0\omega_r$.

12. (M) Derive an expression for a mechanical oscillator that is equivalent to Eq. (I.17). What corresponds physically to the driving current? Draw a corresponding picture.

13. (M) The expression for A_D^2 that we derived, given in Eq. (I.28), is proportional to the Lorentzian function,

$$f_L(\omega) = \frac{1}{\pi} \frac{\gamma}{(\omega - \omega_r)^2 + \gamma^2}.$$

(a) Show that $\int_{-\infty}^{\infty} f_L(\omega) \, d\omega = 1$.

(b) Use MATLAB to plot $L(\omega)$ vs. $\omega - \omega_r$, assuming that $\gamma\omega_r = 1/100$, $1/10\sqrt{10}$, and $1/10$.

(c) We stated that the expression for A_D in Eq. (I.28) is a good approximation to the exact expression for A_D in Eq. (I.21). Compare these expressions by plotting $a(\omega) = (2\gamma/\omega_r)A_D(\omega)$ vs. $x = (\omega - \omega_r)/\gamma$ for both these expressions and for $\gamma/\omega_r = 1/2$, $1/10$, and $1/100$ on both linear and log-log plots, as in oscillatorI7. What are the slopes in the log-log plots for $0 < x < 1$ and $1 < x < \infty$?

14. (M) For our example mechanical oscillator, with the value of α given by $\gamma/\omega_r = 1/10$, find the matrix A that corresponds to Eq. (I.39). Find the eigenvalues and eigenfunctions of this matrix using MATLAB. Show the MATLAB ouput. Assume you have a driving force with an amplitude of 1 mN at the resonant frequency, find the matrix B, and use MATLAB to find $\tilde{u} = [\tilde{x}, \tilde{v}]^t$. Show the MATLAB output. Write the real expressions x and v and use a modified version of OscillatorI4 to draw a phase plot of the oscillation. How much energy is stored in the oscillator?

15. (M and I) MATLAB can work with symbols as well as numbers. Use this capability to obtain Eq. (I.24) and a symbolic expression for $\tilde{u}$. Compare to Eq. (I.19).

16. (E) Obtain a kit for a mechanical oscillator with springs and weights. Measure the oscillations as a function of time and compare to what is theoretically expected.

17. (E) Obtain a kit that will enable you to build an *RLC* oscillator. Measure the oscillations as a function of time and compare to what is experimentally expected.

```
% OscillatorI2
% This program plots the voltage and current in an LC
% LC circuit
% (a) as a function of time, (b) as a phase plot

% Lin = inductance (in mH)
% Cap = capacitance (in pF)
% Vamp = voltage amplitude (in volts)
% phi0 = phase offset

% Vplot = Voltage amplitudes for phase plot
% Nplot = number of phase plots

% Nosc = number of oscillations plotted
% Npoints = number of points in each period

% omega = radial oscillation frequency (in kHz)
% Iamp = current amplitude (in uA)

close all %this clears previous figures
Lin = 1, Cap = 100, Vamp = 1; phi0 = 0,
L = Lin*1e-3; C = Cap*1e-12; om = 1/sqrt(L*C);
Imp = sqrt(L/C); Ia = Vamp/Imp;
% fundamental quantities in SI units

omega = 1e-6*om, % convert frequency to MHz.
Iamp = 1e6*Ia, % convert current amplitude to uA
Nosc = 5, Npoints = 100, Ntot = Nosc*Npoints + 1;
% set the plotting parameters

Nt = 1:Ntot; time = 2*pi*(Nt-1)/Npoints;
tphys = time/omega; % physical time
% set up the time arrays

Voltage = Vamp*cos(time + phi0);
Current = -Iamp*sin(time + phi0);
% voltage and current as functions of time

subplot(2,1,1)
plot(tphys,Voltage,'linewidth',3,'color','r')
set(gca,'fontname','Times New Roman','fontsize',20);
set(gca,'LineWidth',1.5,'TickLength',[0.025 0.025]);
title('Voltage','fontname','Times New Roman','fontsize',20)
xlabel('time (\mus)','fontname','Times New Roman','fontsize',20);
ylabel('Voltage (V)','fontname','Times New Roman','fontsize',20);
```

```matlab
axis tight;
xlim([0 10])
ylim([-1 1])
set(gcf, 'WindowStyle', 'normal');
set(gca, 'Unit', 'inches');
set(gca, 'TickLabelInterpreter','LaTeX',...
'YTick',[-1:1:1],'YTickLabel','$-1$','$0$','$1$')
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])

subplot(2,1,2)
plot(tphys,.001*Current,'linewidth',3,'color','r')
set(gca,'fontname','Times New Roman','fontsize',20);
set(gca,'LineWidth',1.5,'TickLength',[0.025 0.025]);
title('Current','fontname','Times New Roman','fontsize',20)
xlabel('time (\mus)','fontname','Times New Roman','fontsize',20);
ylabel('Current (mA)','fontname','Times New Roman','fontsize',20);
axis tight;
xlim([0 10])
ylim([-.500 .500])
set(gcf, 'WindowStyle', 'normal');
set(gca, 'Unit', 'inches');
set(gca, 'TickLabelInterpreter','LaTeX',...
'YTick',[-.500:.500:.500],'YTickLabel','$-0.5$','$0$','$0.5$')
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])
saveas(gcf,'1.pdf','pdf')

Vplot=[0.5 0.7 0.9], Nplot=3,
% Set the values for the phase plots
%figure
%axis square
close all
for Np=1:Nplot
Vpl=Vplot(Np)*Voltage(1:Ntot);
Ipl=.001*Vplot(Np)*Current(1:Ntot);
plot(Ipl,Vpl,'linewidth',3)
hold on

end
hold off
annotation('arrow','Position',[0.739,0.5,0,0.005])
% text(150.0,0.8,'V_0=0.9','fontname','Times New
Roman','fontsize',16)
annotation('arrow','Position',[0.689,0.5,0,0.005])
% text(85.0,0.7,'V_0=0.7','fontname','Times New
Roman','fontsize',16)
annotation('arrow','Position',[0.640,0.5,0,0.005])
```

```matlab
% text(10.0,0.55,'V_0=0.5','fontname','Times New
Roman','fontsize',16)
set(gca,'fontname','Times New Roman','fontsize',20);
set(gca,'LineWidth',1.5,'TickLength',[0.025 0.025]);
title('Phase plot: Voltage vs. Current','fontname','Times New
Roman','fontsize',20)
xlabel('Current (mA)','fontname','Times New Roman','fontsize',20)
ylabel ('Voltage (V)','fontname','Times New Roman','fontsize',20)
xlim([-.500 .500]) %define limits for x axis
ylim([-1 1]) %define limits for y axis
set(gcf, 'WindowStyle', 'normal');
set(gca, 'Unit', 'inches');
set(gca, 'TickLabelInterpreter','LaTeX',...
'YTick',[-1:.5:1],'YTickLabel','$-1$','$-0.5$','$0$','$0.5$','$1$')

set(gca, 'TickLabelInterpreter','LaTeX',...
'XTick',[-.5:.25:.5],'XTickLabel','$-0.5$','$-0.25$','$0$','$0.25$',
'$0.5$') %define the ticks and use LaTeX interpreter to have real
minus signs
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])
%define the paper size and position
lgd = legend('$V_0=0.5$ V','$V_0=0.7$ V','$V_0=0.9$ V');
%add the legend
% legend('Location','northwest')
set(lgd,'interpreter','latex','FontSize',17);
% set(lgd,'interpreter','latex');
legend ('boxoff'); %turn the box around the elegend off
saveas(gcf,'2.pdf','pdf') %save file as a .pdf file
```

```
% OscillatorI4
% This routine calculates phase diagrams for the parallel
% RLC circuit
% (a) without a driving voltage, (b) with a driving voltage

% Lin = inductance (in mH)
% Cap = capacitance (in pF)
% Res = resistance (in kOhms)

% V0a = initial voltage amplitude, part (a) (in volts)
% I0a = initial current amplitude, part (a) (in uA)

% FreqRat = ratio of driving to resonant frequency,
% part (b)
% V0b = initial voltage amplitudes, part (b) (in volts)
% J0 = driving current amplitude, part (b) (in uA)

% Nosc = number of oscillations plotted
% Npoints = number of points in each period

% omega = natural oscillation frequency
% gamma = damping rate
% Imp = real impedance
% phoff = phase offset

% omr = resonant frequency
% gamRat = ratio of the damping and resonant frequencies
% omRat = ratio of the resonant and natural frequencies
% ADc = driving voltage amplitude;
% CDc = cosine (driving angle)
% SDc = sine (driving angle)
% VDc, VDs = in-phase and quadrature quadratures of
% the driving voltages

% [Current,Voltage] = current and voltage, part (a)

% Part (b):
% [CurD,VolD] = driven current and voltage
% [CurT1,VolT1] = transient current and voltage 1
% [CurT2,VolT2] = transient current and voltage 2

close all

Lin = 1, Cap = 100, Res = 50, V0a = 1,
FreqRat = 1.0, V0b = [0 1], J0 = 10,
% set initial values
```

```matlab
L = Lin*1e-3; C = Cap*1e-12; R = Res*1e3;
% convert to SI units
gamma = 1/(2*R*C); omega=((1/(L*C)) - gamma^2)^(1/2);
% compute the damping coefficient and frequency
gam = gamma/omega; %compute the damping ratio
Imp = sqrt(L/C); phoff = atan(gamma/omega);
% calculate the impedance amplitude and phase offset
Ia = V0a/Imp; I0a = 1e6*Ia;
% compute the initial current amplitude (part a) in uA

Jb = J0*1e-6 %convert to SI units
omr = 1/sqrt(L*C); gamRat = gamma/omr;
omRat = omr/omega;
% compute the resonant frequency, damping ratio,
% and resonant to natural frequency ratio
% (part b)
ADc = 1/((1-FreqRat^2)^2 + 4*FreqRat^2*gamRat^2)^(1/2);
CDc = 2*FreqRat*gamRat*ADc; SDc = (1-FreqRat^2)*ADc;
VDc = FreqRat*Imp*Jb*ADc*CDc;
VDs = FreqRat*Imp*Jb*ADc*SDc;
% in-phase and quadrature components of the driven
% voltage
Vc1 = V0b(1) - VDc; Vc2 = V0b(2) - VDc;
Vs1 = omRat*VDs - gam*Vc1;
Vs2 = omRat*VDs - gam*Vc2;
% transient voltage and current amplitude coefficients

Nosc = 15, Npoints = 100, Ntot = Nosc*Npoints + 1;
% set the plotting parameters

Nt = 1:Ntot; time = 2*pi*(Nt-1)/Npoints;
% set up the time array

% Part (a): Compute and plot voltage and current
Voltage = V0a*cos(time).*exp(-gam*time);
Current = -I0a*sin(time + phoff).*exp(-gam*time);
% voltage and current as functions of time

plot(.001*Current,Voltage,'linewidth',1,'color','r')
% axis([-300,300,-1.0,1.0])
% title('Phase plot: Voltage vs. Current')
% xlabel('Current (\muA)')
% ylabel ('Voltage (V)')
annotation('arrow','Position',[0.691,0.5,0,0.005])
set(gca,'fontname','Times New Roman','fontsize',20);
```

```matlab
set(gca,'LineWidth',1.5,'TickLength',[0.025 0.025]);
title('Phase plot: Voltage vs. Current','fontname','Times New
Roman','fontsize',20)
xlabel('Current (mA)','fontname','Times New Roman','fontsize',20)
ylabel ('Voltage (V)','fontname','Times New Roman','fontsize',20)
xlim([-.500 .500]) %define limits for x axis
ylim([-1 1]) %define limits for y axis
set(gcf, 'WindowStyle', 'normal');
set(gca, 'Unit', 'inches');
set(gca, 'TickLabelInterpreter','LaTeX',...
'YTick',[-1:.5:1],'YTickLabel','$-1$','$-0.5$','$0$','$0.5$','$1$')

set(gca, 'TickLabelInterpreter','LaTeX',...
'XTick',[-.5:.25:.5],'XTickLabel','$-0.5$','$-0.25$','$0$','$0.25$',
'$0.5$') %define the ticks and use LaTeX interpreter to have real
minus signs
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])
%define the paper size and position
saveas(gcf,'3.pdf','pdf') %save file as a .pdf file



% Part (b): Compute and plot the voltage and current
% evolution, starting from the given initial conditions.
% The initial current is assumed to equal zero.

FreqD = FreqRat*omRat; % driving frequency
VolD = VDc*cos(FreqD*time) - VDs*sin(FreqD*time);
CurD = -(1/(omRat*Imp))*(VDs*cos(FreqD*time)...
+ VDc*sin(FreqD*time)); CurD = CurD*1e6
VolT1 = (Vc1*cos(time) - Vs1*sin(time)).*exp(-gam*time);
CurT1 = -(1/(omRat*Imp))*((Vc1-gamRat*Vs1)*sin(time)...
+(Vs1+gamRat*Vc1)*cos(time)).*exp(-gam*time);
CurT1 = CurT1*1e6
VolT2 = (Vc2*cos(time) - Vs2*sin(time)).*exp(-gam*time);
CurT2 = -(1/(omRat*Imp))*((Vc2-gamRat*Vs2)*sin(time)...
+(Vs2+gamRat*Vc2)*cos(time)).*exp(-gam*time);
CurT2 = CurT2*1e6
VolT1 = VolT1 + VolD; CurT1 = CurT1 + CurD;
VolT2 = VolT2 + VolD; CurT2 = CurT2 + CurD;
close all
plot(.001*CurD,VolD,'LineWidth',4)
hold on
plot(.001*CurT1,VolT1,'k',.001*CurT2,VolT2)
set(gca,'fontname','Times New Roman','fontsize',20);
set(gca,'LineWidth',1.5,'TickLength',[0.025 0.025]);
title('Phase plot: Voltage vs. Current','fontname','Times New
```

```matlab
Roman','fontsize',20)
xlabel('Current (mA)','fontname','Times New Roman','fontsize',20)
ylabel ('Voltage (V)','fontname','Times New Roman','fontsize',20)
xlim([-.500 .500]) %define limits for x axis
ylim([-1 1]) %define limits for y axis
set(gcf, 'WindowStyle', 'normal');
set(gca, 'Unit', 'inches');
set(gca, 'TickLabelInterpreter','LaTeX',...
'YTick',[-1:.5:1],'YTickLabel','$-1$','$-0.5$','$0$','$0.5$','$1$')

set(gca, 'TickLabelInterpreter','LaTeX',...
'XTick',[-.5:.25:.5],'XTickLabel','$-0.5$','$-0.25$','$0$','$0.25$',
'$0.5$') %define the ticks and use LaTeX interpreter to have real
minus signs
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])
%define the paper size and position
saveas(gcf,'4.pdf','pdf') %save file as a .pdf file
```

```matlab
% OscillatorI6
% This program plots the FWHM of the Lorentz(ian) function as
% well as the Lorentz(ian) function for positive values in X in a
loglog plot.
% Where A_D^2 is the Eq. I.28 in the text
% Y = 4*A_D^2*gamma^2/omega_r^2
% X = (omega-omega_r)^2/gamma^2

X = -10:.0001:10;
Y = 1./((X).^2+1);
close all %this clears previous figures
plot(X,Y,'linewidth',3,'color','r')
set(gca,'fontname','Times New Roman','fontsize',20);
set(gca,'LineWidth',1.5,'TickLength',[0.03 0.025]);
xlabel('$(\omega-\rm\omega_r)/\gamma$','Interpreter','LaTeX','fontname',
'Times New Roman','fontsize',20)
ylabel ('4\it A\rm_D^2 \gamma^2/\omega_r^2','fontname','Times New
Roman','fontsize',20)
xlim([-10 10]) %define limits for x axis
ylim([0 1.2]) %define limits for y axis
set(gcf, 'WindowStyle', 'normal'); set(gca, 'Unit',
'inches'); set(gca, 'TickLabelInterpreter','LaTeX',...
'XTick',[-10:5:10],'XTickLabel','$-10$','$-5$','$0$','$5$','$10$')
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])
%define the paper size and position
annotation('doublearrow',[.48 .555],[.48 .48])
text(-.5,0.45,'2\gamma','fontname','Times New
Roman','fontsize',20)
saveas(gcf,'6a.pdf','pdf') %save file as 6a.pdf file

X = .01:.0001:100;
Y = 1./(X.^2+1);
close all %this clears previous figures
loglog(X,Y,'linewidth',3,'color','r')
grid on
set(gca,'fontname','Times New Roman','fontsize',20);
set(gca,'LineWidth',1.5,'TickLength',[0.03 0.025]);
xlabel('$(\omega-\rm\omega_r)/\gamma$','Interpreter','LaTeX','fontname',
'Times New Roman','fontsize',20)
ylabel ('4\it A\rm_D^2 \gamma^2/\omega_r^2','fontname','Times New
Roman','fontsize',20)
xlim([0.01 100]) %define limits for x axis
ylim([0.0001 10]) %define limits for y axis
set(gcf, 'WindowStyle', 'normal');
set(gca, 'Unit', 'inches');
set(gca, 'TickLabelInterpreter','LaTeX',...
```

```
'XTick',[0.01 0.1 1 10 100])
set(gcf, 'PaperPosition',1.9*[0 0 4 3],'PaperSize',1.9*[4 3])
%define the paper size and position
% annotation('doublearrow',[.48 .555],[.48 .48])
% text(-.5,0.45,'2\gamma','fontname','Times New
Roman','fontsize',20)
saveas(gcf,'6b.pdf','pdf') %save file as a .pdf file
```